AIL 722: Reinforcement Learning

Lec 2: Hidden Markov Models (Part 1: Likelihood)

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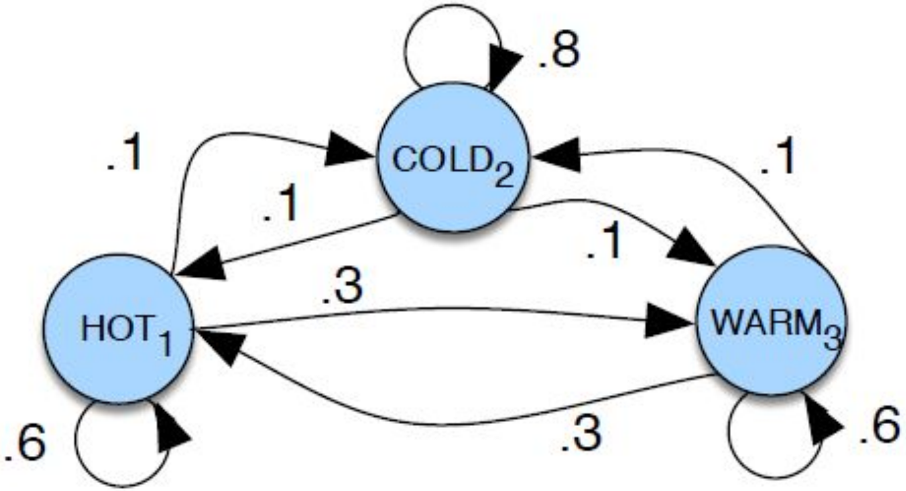
Why HMMs?

- Uncertainty: Zone of probabilistic reasoning
- Foundational material towards MDPs
- Constructs: Sequences of states, a.k.a. trajectories
- Algorithms: Iterative approaches
- Using observed data to make inferences

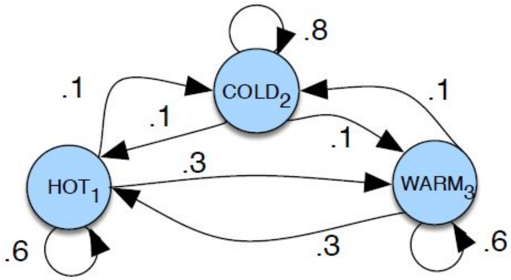
Markov Chain



Weather

Source: [SLP, Dan Jurafsky]

Markov Chain



$$\mathcal{S} = \{s_1, s_2, \ldots, s_N\}$$ A set of N states

$$T = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{pmatrix}$$ A transition probability matrix

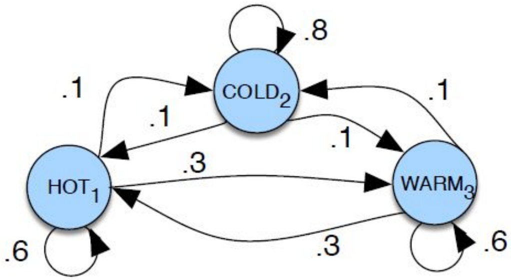
Markov Chain

$$\rho = \{p(s_1), p(s_2), \ldots, p(s_N)\}$$ Initial state distribution

$$p(s_i = a \mid s_1, s_2, \ldots, s_{i-1}) = p(s_i = a \mid s_{i-1})$$

Markov Property

Exercise



➔ Compute the probability of the sequences
- ◆ Cold, Cold, Cold, Cold
- ◆ Cold, Hot, Cold, Hot

What information is missing from this question?

Initial State Distribution

Hidden Markov Model

$$\mathcal{S} = \{s_1, s_2, \ldots, s_N\}$$

$$T = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{pmatrix}$$

$$\rho = \{p(s_1), p(s_2), \ldots, p(s_N)\}$$

$\mathcal{O} = \{o_1, o_2, \ldots, o_M\}$ A set of M possible observations

$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1M} \\ b_{21} & b_{22} & \cdots & b_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N1} & b_{N2} & \cdots & b_{NM} \end{pmatrix}$ Observation probability matrix, where $b_{ij} = p(o_j \mid s_i)$

$p(o_i \mid s_1, \ldots, s_i, \ldots, s_T, o_1, \ldots, o_i, \ldots, o_T) = p(o_i \mid s_i)$ **Output Independence**

Hidden Markov Model

$$O = \{o_1, o_2, \ldots, o_T\}$$

Input to the HMM: A sequence of T observations

$$\mathcal{S} = \{s_1, s_2, \ldots, s_N\}$$

$$T = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{pmatrix}$$

$$\rho = \{p(s_1), p(s_2), \ldots, p(s_N)\}$$

$$\mathcal{O} = \{o_1, o_2, \ldots, o_M\}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1M} \\ b_{21} & b_{22} & \ldots & b_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N1} & b_{N2} & \ldots & b_{NM} \end{pmatrix}$$

Where are HMMs used?

Speech Recognition	Acoustic Signal	Phenomes: Pat/Bat
Activity Recognition	Sensor Readings	Activity: walking
Music Transcription	Audio features	Musical notes
Finance	Financial indicators	Bull, Bear, Stable

HMM: Three Fundamental Problems

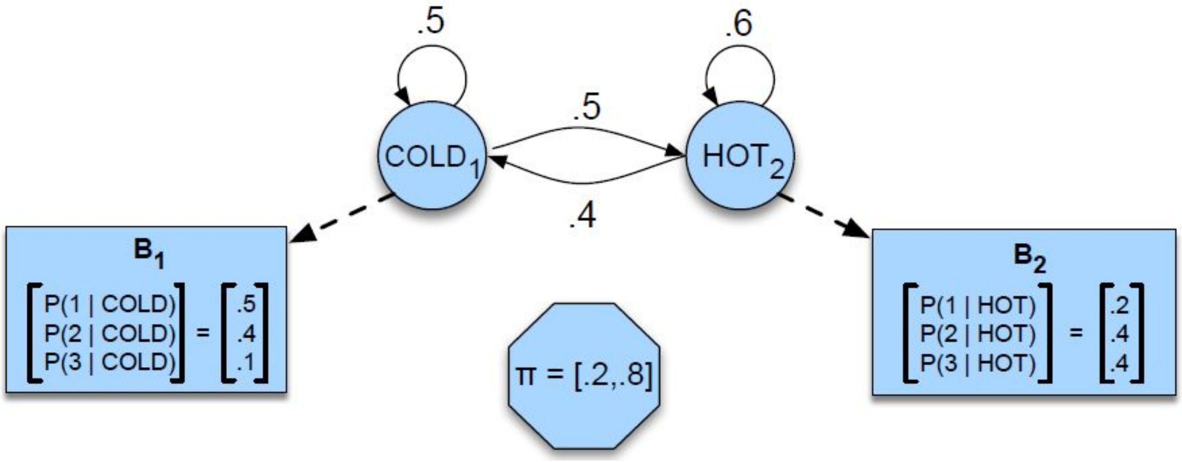
Problem 1 (Likelihood):
Given an HMM $\lambda = (\mathcal{T}, \mathcal{B})$ and an observation sequence O, determine the likelihood $P(O \mid \lambda)$.

Problem 2 (Decoding):
Given an observation sequence O and an HMM $\lambda = (\mathcal{T}, \mathcal{B})$, discover the best hidden state sequence.

Problem 3 (Learning):
Given an observation sequence O and the set of states in the HMM, learn the HMM parameters $\mathcal{T}$ and $\mathcal{B}$.

Running Example: Weather and Ice Cream

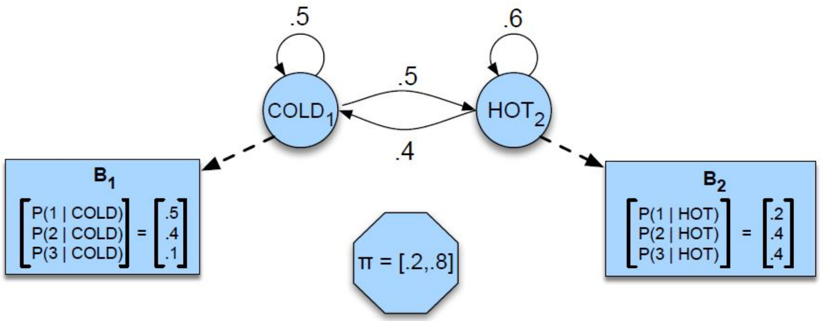
- Given a sequence of observations O (each an integer representing the number of ice creams eaten on a given day) find the 'hidden' sequence of weather states (H or C)



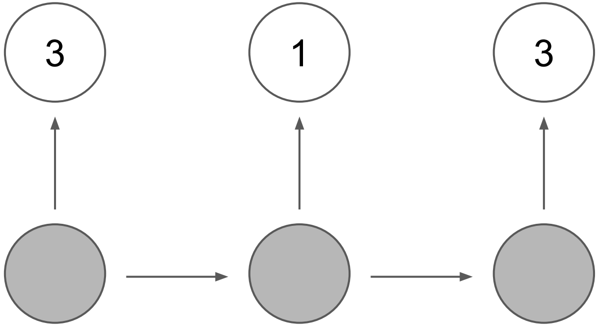
Example from SLP, Dan Jurafsky

Problem 1: Likelihood Computation

Likelihood Computation

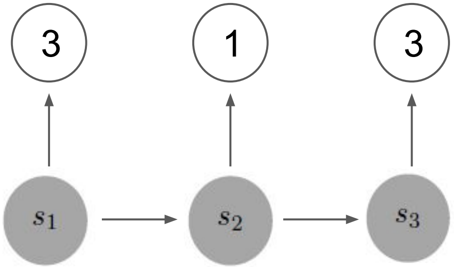


- Given the ice-cream eating HMM, what is the probability of the sequence of ice creams eaten being 3, 1, 3?
  - 3 ice creams on day 1, 1 on day 2, and 3 on day 3



$$p(3,1,3)?$$

Marginalise over Hidden State Seq.



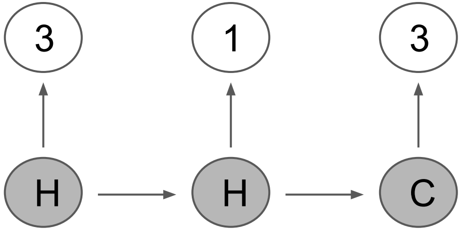
$$p(O) = p(o_1, o_2, \ldots, o_T) \qquad p(S) = p(s_1, s_2, \ldots, s_T)$$

$$p(O) = \sum_S p(O, S)$$

$$= \sum_S p(O \mid S) \cdot p(S)$$

$$p(O \mid S) = \prod_{i=1}^{T} p(o_i \mid s_i)$$

Known Hidden State Seq.



$$p(O \mid S) = \prod_{i=1}^{T} p(o_i \mid s_i)$$

$$p(3, 1, 3 \mid H, H, C)$$

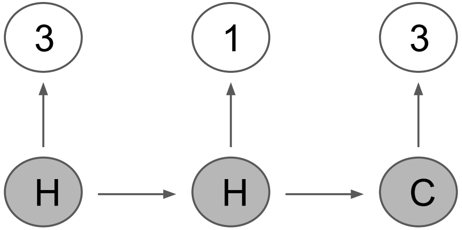
$$= p(o_1 = 3, o_2 = 1, o_3 = 3 \mid s_1 = H, s_2 = H, s_3 = C)$$

$$= p(o_1 = 3 \mid s_1 = H, s_2 = H, s_3 = C) \cdot p(o_2 = 1, o_3 = 3 \mid o_1 = 3, s_1 = H, s_2 = H, s_3 = C)$$

$$= p(o_1 = 3 \mid s_1 = H) \; p(o_2 = 1, o_3 = 3 \mid o_1 = 3, s_1 = H, s_2 = H, s_3 = C)$$

$$p(3, 1, 3 \mid H, H, C) = p(3 \mid H) \cdot p(1 \mid H) \cdot p(3 \mid C)$$

Marginalise via Enumeration



$$p(O) = \sum_S p(O, S)$$

$$p(3,1,3) = p(3,1,3,C,C,C) + p(3,1,3,C,C,H) + p(3,1,3,C,H,H) + \dots$$

$$= p(3,1,3 \mid C,C,C) \cdot p(C,C,C) + p(3,1,3 \mid C,C,H) \cdot p(C,C,H) + \dots$$

$$p(3,1,3 \mid C,C,C) = p(3 \mid C) \cdot p(1 \mid C) \cdot p(3 \mid C)$$

$$p(C,C,C) = p(s_i = C) \cdot p(C \mid C) \cdot p(C \mid C)$$

Brute Force Enumeration

Algorithm 1 Brute Force Likelihood

1: Enumerate all possible hidden state sequences (N^T of them)
2: **for** each hidden state sequence **do**
3: Compute $p(o_{1:T} \mid s_{1:T})p(s_{1:T})$
4: **end for**
5: Add up all the above obtained $p(o_{1:T}, s_{1:T})$ to marginalize over all possible hidden state sequences

Do you see a problem with this approach?

Forward Algorithm

$$\alpha_t(j) = p(o_1, o_2, \ldots, o_t, s_t = j)$$

1. Initialization:

$$\alpha_1(j) = p(s_1 = j) \cdot p(o_1 \mid s_1 = j) \qquad 1 \leq j \leq N$$

2. Recursion:

$$\alpha_t(j) = \sum_{i=1}^{N} \alpha_{t-1}(i) \cdot p(s_t = j \mid s_{t-1} = i) \cdot p(o_t \mid s_t = j) \qquad 1 \leq j \leq N, 1 < t \leq T$$

3. Termination:

$$p(O) = \sum_{i=1}^{N} \alpha_T(i) \qquad \sum_{i=1}^{N} p(o_1, \ldots, o_T, s_T = i)$$

Marginalise over final state

Forward Algo: Rationale

$$\alpha_t(j) = p(o_1, o_2, \ldots, o_t, s_t = j)$$

$$= \sum_{i=1}^{N} p(o_1, o_2, \ldots, o_{t-1}, s_{t-1} = i, o_t, s_t = j)$$

$$= \sum_{i=1}^{N} p(o_{1:t-1}, s_{t-1} = i) \cdot p(o_t, s_t = j \mid o_{1:t-1}, s_{t-1} = i)$$

$$= \sum_{i=1}^{N} p(o_{1:t-1}, s_{t-1} = i) \cdot p(s_t = j \mid o_{1:t-1}, s_{t-1} = i) \cdot p(o_t \mid o_{1:t-1}, s_{t-1} = i, s_t = j)$$

$$= \sum_{i=1}^{N} p(o_{1:t-1}, s_{t-1} = i) \cdot p(o_t \mid s_t = j) \cdot p(s_t = j \mid s_{t-1} = i)$$

$$\alpha_t(j) = \sum_{i=1}^{N} \alpha_{t-1}(i) \cdot p(s_t = j \mid s_{t-1} = i) \cdot p(o_t \mid s_t = j)$$